

Solutions to degenerate complex Hessian equations

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Abstract

Let (X, ω) be a n -dimensional compact Kähler manifold. In this paper we study degenerate complex Hessian equations of the form $(\omega + dd^c \varphi)^m \wedge \omega^{n-m} = F(x, \varphi) \omega^n$. We develop the first steps of a potential theory for the associated complex Hessian operator. In particular we define a notion of bounded weak solution for this equation. We then prove that under some natural conditions on F and a curvature assumption on ω , this equation has a unique continuous solution.

1 Introduction

Let (X, ω) be a compact Kähler manifold of complex dimension n . Fix an integer m between 1 and n , and let d, d^c denote the usual real differential operators $d := \partial + \bar{\partial}$, $d^c = \frac{\sqrt{-1}}{2\pi}(\bar{\partial} - \partial)$ so that $dd^c = \frac{i}{\pi} \partial \bar{\partial}$.

We are studying degenerate complex Hessian equations of the form

$$(1.1) \quad (\omega + dd^c \varphi)^m \wedge \omega^{n-m} = F(x, \varphi) \omega^n,$$

where the density $F : X \times \mathbb{R} \rightarrow \mathbb{R}^+$ satisfies some natural integrability conditions (see the theorem below).

The case $m = 1$ corresponds to the Laplace equation and the case $m = n$ corresponds to degenerate complex Monge-Ampère equations which has been studied intensively in the recent years (see [3, 4, 6, 7, 8, 9, 10, 11, 12, 18, 19, 22, 23, 24, 31, 32, 33, 34]). So equation (1.1) is a generalization of both Laplace and Monge-Ampère equations.

The non degenerate complex Hessian equation on compact Kähler manifold, where $F(x, \varphi) = f(x)$, with $0 < f \in \mathcal{C}^\infty(X)$, has been studied recently in [25, 26, 28]. In [25] and [28], the authors independently solved this equation with a strong additional hypothesis, assuming (X, ω) has non negative holomorphic bisectional curvature. Later on, in [26] an a priori \mathcal{C}^2 estimate was obtained without curvature assumption. The general problem is still open.

In this paper we develop the first steps for a potential theory for the complex Hessian equation on compact Kähler manifold. Following Blocki [3] we define the class of (ω, m) -subharmonic functions which is a generalization of the class of ω -plurisubharmonic functions when $m = n$. Since we do not know at this moment if any (ω, m) -subharmonic function is a decreasing limit of a sequence of smooth ones, the definition of the complex Hessian operator on bounded (ω, m) -subharmonic functions is delicate.

In order to extend the definition of the complex Hessian operator to bounded non smooth (ω, m) -subharmonic functions, we introduce a capacity adapted to the class of (ω, m) -subharmonic functions and use it to define the concept of quasi-uniform convergence. This allows us to define a suitable class of bounded and quasi-continuous (ω, m) -subharmonic functions on which the complex Hessian operator is well defined and continuous under quasi-uniform convergence. We show that this definition coincides with the definition in the spirit of Bedford and Taylor method for the complex Monge-Ampère operator. We prove a comparison principle and convergence results for this operator. Then we consider the degenerate Hessian equation (1.1) when the right hand side $F(x, \cdot) = f(x)$ is in $L^p(X)$.

Using Moser iteration technique and the result in [25, 28] we prove that there exists a bounded weak solution to this equation when $p > n$, and (X, ω) has non negative holomorphic bisectional curvature. The continuity of the solution requires uniform stability estimates which were obtained recently by Kolodziej and Dinew [17] under the more general condition $p > n/m$. The key point in these estimates is a domination between volume and capacity as in the case of complex Monge-Ampère equations [11], [18], [31].

The main result of this paper is the following:

Theorem. *Let (X, ω) be a n -dimensional compact Kähler manifold of non negative holomorphic bisectional curvature. Fix $1 \leq m \leq n$ and $p > n/m$. Let $F : X \times \mathbb{R} \rightarrow [0, +\infty)$ be a function satisfying the following conditions:*

- (F1) For all $x \in X$, $t \mapsto F(x, t)$ is non-decreasing and continuous,*
- (F2) For all $t \in \mathbb{R}$, $x \mapsto F(x, t)$ is in $L^p(X)$,*
- (F3) There exists $t_0 \in \mathbb{R}$ such that $\int_X F(\cdot, t_0) \omega^n = \int_X \omega^n$.*

Then there exists a function $\varphi \in \mathcal{P}_m(X, \omega) \cap \mathcal{C}^0(X)$, unique up to an additive constant, such that

$$(\omega + dd^c \varphi)^m \wedge \omega^{n-m} = F(x, \varphi) \omega^n.$$

Moreover if $\forall x \in X, t \mapsto F(x, t)$ is increasing, then the solution is unique.

Remark 1.1. The curvature assumption can be relaxed right after the non degenerate complex Hessian equation is solved in full generality.

Note that the condition (F3) is automatically satisfied if $F(., -\infty) = 0$ and $F(., +\infty) = +\infty$. An important particular case is the exponential function $F(x, t) = f(x)e^t$.

A particular case of this result has been obtained in [17]. The key point in their proof is a domination between volume and capacity. Our main result is proved using this technique and the recent result in the smooth case [25, 28].

Acknowledgement. The paper is part of my Ph.D Thesis. I would like to express my great gratitude to my advisor, Professor Ahmed Zeriahi, for sacrificing his very valuable time for me. I wish to express my sincere gratitude to Professor Vincent Guedj for his very useful suggestions and discussions to improve the paper. I also wish to say a special word of thanks to Sébastien Boucksom for his kind invitation to IMJ and useful discussions. This paper owes much to their help and constant encouragement.

2 Preliminaries

In this section we introduce the notion of (ω, m) -subharmonic functions. We define a suitable class of bounded (ω, m) -subharmonic functions on which the complex Hessian operator is well-defined, continuous under suitable convergence and satisfies the comparison principle.

2.1 Elementary symmetric functions

First, we recall some basic properties of elementary symmetric functions (see [3, 14, 20]). We use the notations in [3]. Let S_k , $k = 1, \dots, n$ be the k -elementary symmetric function, that is, for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$,

$$S_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}.$$

For convenience, we set $S_0(\lambda) = 1$ and $S_k(\lambda) = 0$ if $k > n$ or $k < 0$. We have the following identity

$$(\lambda_1 + t) \dots (\lambda_n + t) = \sum_{k=0}^n S_k(\lambda) t^{n-k}, \quad t \in \mathbb{R}.$$

We denote Γ_k the closure of the connected component of $\{S_k(\lambda) > 0\}$ containing $(1, \dots, 1)$. We can show that (see [20])

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_k(\lambda_1 + t, \dots, \lambda_n + t) \geq 0, \quad \forall t \geq 0\}.$$

Thus it follows from the identity

$$S_m(\lambda_1 + t, \dots, \lambda_n + t) = \sum_{k=0}^m \binom{n-k}{m-k} S_k(\lambda) t^{m-k}, \quad t \in \mathbb{R}$$

that

$$\Gamma_k := \{\lambda \in \mathbb{R}^n : S_j(\lambda) \geq 0, \quad \forall 1 \leq j \leq k\}.$$

We have an obvious inclusion

$$\Gamma_n \subset \dots \subset \Gamma_1.$$

By Gårding [20] the set Γ_k is a convex cone in \mathbb{R}^n and $S_k^{1/k}$ is concave on Γ_k . For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and $1 \leq j \leq n$ we define

$$S_{k,j}(\lambda) := S_k(\lambda^{(j)}),$$

where $\lambda^{(j)} = (\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_n) \in \mathbb{R}^{n-1}$. It is easy to check that

$$(2.1) \quad \sum_{j=1}^n S_{k,j}(\lambda) = (n-k) S_k(\lambda), \quad \forall \lambda \in \mathbb{R}^n.$$

We denote by \mathcal{H} the vector space (over \mathbb{R}) of complex hermitian $n \times n$ matrices. For $A \in \mathcal{H}$ we denote $\lambda(A) = (\lambda_1, \dots, \lambda_n)$ the eigenvalues of A . We set

$$\tilde{S}_k(A) = S_k(\lambda(A)).$$

From the identity

$$\det(A + tI) = \sum_{k=0}^n \tilde{S}_k(A) t^{n-k}, \quad t \in \mathbb{R}$$

it follows that the function \tilde{S}_k can be seen as the sum of all principal minors of order k ,

$$\tilde{S}_k(A) = \sum_{|I|=k} A_{II}.$$

Thus \tilde{S}_k is a homogeneous polynomial of order k on \mathcal{H} which is hyperbolic with respect to the identity matrix I (that is for every $A \in \tilde{S}$ the equation $\tilde{S}_k(A + tI) = 0$ has n real roots; see [20]). As in [20] (see also [3]) we define the cone

$$\tilde{\Gamma}_k := \{A \in \mathcal{H} : \tilde{S}_k(A + tI) \geq 0, \forall t \geq 0\}.$$

We have

$$\tilde{\Gamma}_k := \{A \in \mathcal{H} : \lambda(A) \in \Gamma_k\}.$$

It follows from [20] that the cone $\tilde{\Gamma}_k$ is convex and the function $\tilde{S}_k^{1/k}$ is concave on $\tilde{\Gamma}_k$.

For $A \in \mathcal{H}$ we define

$$D_k(A) = \left[\frac{\partial \tilde{S}_k}{\partial a_{p\bar{q}}}(A) \right] \in \mathcal{H}.$$

By Euler's identity for homogeneous functions one has

$$(2.2) \quad \sum_{p,q=1}^n \frac{\partial \tilde{S}_k}{\partial a_{p\bar{q}}}(A) a_{p\bar{q}} = k \tilde{S}_k(A).$$

Lemma 2.1. [3] *Let $A \in \tilde{\Gamma}_k$ with $\lambda = \lambda(A)$. Then the eigenvalues of $D_k(A)$ are given by*

$$\lambda(D_k(A)) = (S_{k-1,1}(\lambda), \dots, S_{k-1,n}(\lambda)).$$

Thus $D_k(A) \in \tilde{\Gamma}_k$ and

$$\text{tr}(D_k(A)) = (n - k + 1) \tilde{S}_{k-1}(A).$$

2.2 ω -subharmonic functions

In this section, we consider $\Omega \subset X$ an open subset contained in a local chart.

Definition 2.2. A function $u \in L^1(\Omega)$ is called weakly ω -subharmonic if

$$dd^c u \wedge \omega^{n-1} \geq 0,$$

in the weak sense of currents.

Thanks to Littman [39] we have the following approximation properties.

Proposition 2.3. *Let u be a weakly ω -subharmonic function in Ω . Then there exists a one parameter family of functions u_h with the following properties: For every compact subset $\Omega' \subset \Omega$*

- a) u_h is smooth in Ω' for h sufficiently large,*
- b) $dd^c u_h \wedge \omega^{n-1} \geq 0$ in Ω' ,*
- c) u_h is non increasing with increasing h , and $\lim_{h \rightarrow \infty} u_h(x) = u(x)$ almost every where in Ω' ,*
- d) u_h is given explicitly as*

$$(2.3) \quad u_h(y) = \int_{\Omega} K_h(x, y) u(x) dx,$$

where K_h is a smooth non negative function and

$$\int_{\Omega} K_h(x, y) dy \rightarrow 1,$$

uniformly in $x \in \Omega'$.

Definition 2.4. A function u is called ω -subharmonic if it is weakly ω -subharmonic and for every $\Omega' \Subset \Omega$, $\lim_{h \rightarrow \infty} u_h(x) = u(x)$, $\forall x \in \Omega'$, where u_h is constructed as in Proposition 2.3.

Remark 2.5. Any continuous weakly ω -subharmonic function is ω -subharmonic.

If (u_j) is a sequence of continuous ω -subharmonic functions decreasing to u and if $u \neq -\infty$ then u is ω -subharmonic.

If u is weakly ω -subharmonic then the pointwise limit of (u_h) is a ω -subharmonic function.

Let (u_j) is a sequence of ω -subharmonic functions and (u_j) is uniformly bounded from above. Then $u := (\limsup_j u_j)^*$ is ω -subharmonic. Where for a function v , we set v^* the upper semicontinuous regularization of v .

The following Hartogs lemma can be proved in the same way as in the case of subharmonic functions.

Lemma 2.6. *Let $u_t(x)$, $t > 0$ be a family of non positive ω -subharmonic functions in Ω and u_t is uniformly bounded in $L^1_{loc}(\Omega)$. Suppose that for compact subset K in Ω there exists a constant C such that $v(x) = [\limsup_{t \rightarrow +\infty} u_t(x)]^* \leq C$ on K . Then for every $\epsilon > 0$, there exists T_ϵ such that $u_t(x) \leq C + \epsilon$ for $t \geq T_\epsilon$ and $x \in K$.*

2.3 (ω, m) -subharmonic functions

We associate real $(1,1)$ -forms α in \mathbb{C}^n with hermitian matrices $[a_{j\bar{k}}]$ by

$$\alpha = \frac{i}{\pi} \sum_{j,k} a_{j\bar{k}} dz_j \wedge d\bar{z}_k.$$

Then the canonical Kähler form β is associated with the identity matrix I . It is easy to see that

$$(2.4) \quad \binom{n}{k} \alpha^k \wedge \beta^{n-k} = \tilde{S}_k(A) \beta^n.$$

Definition 2.7. Let α be a real $(1,1)$ -form on X . We say that α is (ω, m) -positive at a given point $P \in X$ if at this point we have

$$\alpha^k \wedge \omega^{n-k} \geq 0, \quad \forall k = 1, \dots, m.$$

We say that α is (ω, m) -positive if it is (ω, m) -positive at any point of X .

Remark 2.8. Locally at $P \in X$ with local coordinates z_1, \dots, z_n , we have

$$\alpha = \frac{i}{\pi} \sum_{j,k} \alpha_{j\bar{k}} dz_j \wedge d\bar{z}_k,$$

and

$$\omega = \frac{i}{\pi} \sum_{j,k} g_{j\bar{k}} dz_j \wedge d\bar{z}_k.$$

Then α is (ω, m) -positive at P if and only if the eigenvalues $\lambda(g^{-1}\alpha) = (\lambda_1, \dots, \lambda_n)$ of the matrix $\alpha_{j\bar{k}}(P)$ with respect to the matrix $g_{j\bar{k}}(P)$ is in Γ_m . These eigenvalues are independent of any choice of local coordinates.

Proposition 2.9. *Let $\alpha \in \Lambda^{1,1}(X)$ be a real $(1,1)$ -form on X . Then α is (ω, m) -positive if and only if*

$$\alpha \wedge \beta_1 \wedge \dots \wedge \beta_{m-1} \wedge \omega^{n-m} \geq 0,$$

for all (ω, m) -positive forms $\beta_1, \dots, \beta_{m-1}$.

Definition 2.10. A current T of bidegree (p, p) is said to be (ω, m) -positive if

$$\alpha_1 \wedge \dots \wedge \alpha_{n-p} \wedge T \geq 0,$$

for all smooth (ω, m) -positive $(1,1)$ -forms α_i .

Following Blocki [5] we can define (ω, m) -subharmonicity for non-smooth functions.

Definition 2.11. A function $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ is called (ω, m) -subharmonic if the following conditions hold

- (i) In any local chart Ω , given ρ the local potential of ω and set $u := \rho + \varphi$, then u is ω -subharmonic,
- (ii) for every smooth (ω, m) -positive forms $\beta_1, \dots, \beta_{m-1}$ we have, in the weak sense of distributions,

$$(\omega + dd^c \varphi) \wedge \beta_1 \wedge \dots \wedge \beta_{m-1} \wedge \omega^{n-m} \geq 0.$$

Let $SH_m(X, \omega)$ be the set of all (ω, m) -subharmonic functions on X . Observe that by definition, any $\varphi \in SH_m(X, \omega)$ is upper semicontinuous.

The following properties of (ω, m) -subharmonic functions are easy to show.

Proposition 2.12. (i) If $\varphi \in C^2(X)$ then φ is (ω, m) -subharmonic if the form $(\omega + dd^c \varphi)$ is (ω, m) -positive.

(ii) If $\varphi, \psi \in SH_m(X, \omega)$ then $\max(\varphi, \psi) \in SH_m(X, \omega)$.

(ii) If $\varphi, \psi \in SH_m(X, \omega)$ and $\lambda \in [0, 1]$ then $\lambda\varphi + (1 - \lambda)\psi \in SH_m(X, \omega)$.

(iii) If $(\varphi_j) \subset SH_m(X, \omega)$ is uniformly bounded from above then

$(\limsup_j \varphi_j)^* \in SH_m(X, \omega)$.

Thanks to Hartogs Lemma 2.6, the following proposition can be proved in the same way as in the case of ω -plurisubharmonic function (see [23]).

Proposition 2.13. Let (φ_j) be a sequence of functions in $SH_m(X, \omega)$.

(i) If (φ_j) is uniformly bounded from above on X , then either φ_j converges uniformly to $-\infty$ or the sequence (φ_j) is relatively compact in $L^1(X)$.

(ii) If $\varphi_j \rightarrow \varphi$ in $L^1(X)$ then

$$\sup_X \varphi = \lim_j \sup_X \varphi_j.$$

The following compactness lemma can be deduced easily from Proposition 2.13.

Lemma 2.14. There exists a constant $C_0 > 0$ such that for all $\varphi \in SH_m(X, \omega)$ satisfying $\sup_X \varphi = 0$ we have

$$\int_X \varphi \omega^n \geq -C_0.$$

It then follows that

$$\mathcal{C} := \{\varphi \in SH_m(X, \omega) / \sup_X \varphi \leq 0; \int_X \varphi \omega^n \geq -C_0\}$$

is a convex compact subset of $L^1(X)$.

2.4 Non degenerate complex Hessian equations

We summarize here some recent results on the non degenerate complex Hessian equation on compact Kähler manifolds,

$$(2.5) \quad (\omega + dd^c \varphi)^m \wedge \omega^{n-m} = f \omega^n,$$

where $0 < f$ is smooth such that

$$(2.6) \quad \int_X f \omega^n = \int_X \omega^n.$$

Hou [25] and Jbilou [28] independently proved in 2008-2009 the following theorem:

Theorem 2.15. *[25, 28] If (X, ω) has non negative holomorphic bisectional curvature and $0 < f \in C^\infty(X)$ satisfies (2.6) then equation (2.5) has a unique (up to an additive constant) smooth solution.*

Later on Hou-Ma-Wu [26] proved a second order a priori estimate (without curvature assumption):

Theorem 2.16. *[26] Let (X, ω) be a n -dimensional compact Kähler manifold. If $0 < f \in C^\infty(X)$ satisfies (2.6) and $\varphi \in C^4(X)$ is a normalized solution to equation (2.5) then we have the following a priori estimate*

$$\sup_X |dd^c \varphi|_\omega \leq C(1 + \sup_X |\nabla \varphi|_\omega^2)$$

where the constant $C > 0$ depends only on a lower bound for the holomorphic bisectional curvature of (X, ω) , $\|f^{1/m}\|_{C^2(X)}$, ω, m, n .

The complex Hessian equation (2.7) in domains of \mathbb{C}^n , i.e. equations of the form

$$(2.7) \quad (dd^c u)^m \wedge \beta^{n-m} = f \beta^n,$$

where β is the canonical Kähler form in \mathbb{C}^n , was considered by Li [37] and Blocki [5]. Existence and uniqueness of smooth solution to the Dirichlet problem in smoothly bounded domains with $(m-1)$ -pseudoconvex boundary was obtained in [37]. In [5], a potential theory for m -subharmonic functions was developed and the corresponding degenerate Dirichlet problem was solved. Recently, Sadullaev and Abdullaev studied capacities and polar sets for m -subharmonic functions [40]. Note that the corresponding problem when β is not the euclidean Kähler form is fully open.

It is important to mention that the study of real Hessian equations is a classical subject which has been developed previously in many papers, for example [13, 14, 27, 35, 36, 41, 43, 44, 46]. In particular, the Dirichlet problem in domains whose boundary satisfies some convexity assumptions has been solved in [13]. The degenerate Hessian equation was then considered in [27] and [44]. A nonlinear potential theory has been developed in [36] and [43]. For a survey of the real Hessian equation theory, we refer the reader to [46].

3 Complex Hessian operators.

In this section we develop the first steps of a pluripotential theory adapted to complex Hessian equations, in order to study the degenerate complex Hessian equation (1.1). It is classical that one can approximate any quasi plurisubharmonic function by decreasing sequence of smooth quasi psh functions [12],[15]. For (ω, m) -subharmonic functions we do not know at this moment if the same regularization property is true. To go around this difficulty we introduce a notion of capacity adapted to (ω, m) -subharmonic functions and define the notion of convergence in capacity.

3.1 Capacity

Definition 3.1. Let $E \in X$ be a Borel subset. We define the inner (ω, m) -capacity of E by

$$\text{cap}_{\omega, m}(E) := \sup \left\{ \int_E \omega_\varphi^m \wedge \omega^{n-m} / \varphi \in SH_m(X, \omega) \cap \mathcal{C}^2(X), 0 \leq \varphi \leq 1 \right\}.$$

The outer (ω, m) -capacity of E is defined to be

$$\text{Cap}_{\omega, m}(E) = \inf \{ \text{cap}_{\omega, m}(U) \mid E \subset U, U \text{ is an open subset of } X \}.$$

It follows directly from the definition that $\text{Cap}_{\omega, m}$ is monotone and σ -sub-additive.

Observe that if $\varphi \in SH_m(X, \omega) \cap \mathcal{C}^2(X)$, $0 \leq \varphi \leq M$ then, for any Borel subset $E \subset X$,

$$(3.1) \quad \int_E \omega_\varphi^m \wedge \omega^{n-m} \leq M^m \text{cap}_{\omega, m}(E).$$

Definition 3.2. A sequence (φ_j) converges in $\text{cap}_{\omega, m}$ to φ if for any $\delta > 0$ we have

$$\lim_{j \rightarrow \infty} \text{cap}_{\omega, m}(|\varphi_j - \varphi| > \delta) = 0.$$

Definition 3.3. A sequence of functions (φ_j) converges quasi-uniformly to φ on X (w.r.t $\text{Cap}_{\omega,m}$) if for every $\epsilon > 0$ there exists an open subset $U \subset X$ such that $\text{Cap}_{\omega,m}(U) \leq \epsilon$ and φ_j converges uniformly to φ in $X \setminus U$.

This convergence is almost equivalent to the convergence in capacity as the following result shows

Proposition 3.4. (i) If φ_j converges quasi-uniformly to φ , then for each $\delta > 0$,

$$\lim_{j \rightarrow \infty} \text{Cap}_{\omega,m}(|\varphi_j - \varphi| > \delta) = 0.$$

(ii) Conversely, assume that (φ_j) is a sequence of functions and φ is a function such that, for every $\delta > 0$,

$$\lim_{j \rightarrow \infty} \text{Cap}_{\omega,m}(|\varphi_j - \varphi| > \delta) = 0.$$

Then there exists a subsequence (φ_{j_k}) converging quasi-uniformly to φ .

The condition (ii) will be called convergence in capacity as in the classical case of potential theory.

Proof. The first part is obvious, so we only prove the second part. We can find a subsequence (and for convenience we still denote it by (φ_j)) such that

$$\text{Cap}_{\omega,m}(|\varphi_j - \varphi| > 1/j) \leq 2^{-j}, \quad \forall j.$$

For each j , let U_j be an open subset of X such that $(|\varphi_j - \varphi| > 1/j) \subset U_j$ and $\text{cap}_{\omega,m}(U_j) \leq 2^{-j+1}$. Then for each $\epsilon > 0$, we can find $k \in \mathbb{N}$ such that $\cup_{j \geq k} U_j$ is the open subset of $\text{Cap}_{\omega,m}$ less than ϵ and φ_j converges uniformly to φ on its complement. \square

Definition 3.5. We denote $\mathcal{P}_m(X, \omega)$ the set of all functions $\varphi \in SH_m(X, \omega)$ such that there exists a sequence of \mathcal{C}^2 , (ω, m) -subharmonic functions (φ_j) converging quasi-uniformly to φ on X . Equivalently, we can replace quasi-uniform convergence by convergence in Capacity thanks to Proposition 3.4

Proposition 3.6. (i) Any $\varphi \in \mathcal{P}_m(X, \omega)$ is quasi continuous, that means, for any $\epsilon > 0$ there exists an open subset $U \subset X$ of $\text{Cap}_{\omega,m}$ less than ϵ such that φ is continuous on $X \setminus U$.

(ii) If $(\varphi_j) \downarrow \varphi$ in $\mathcal{P}_m(X, \omega)$ then (φ_j) converges quasi-uniformly to φ .

Proof. The first statement follows directly from the definition. From (i), for each $\epsilon > 0$, there exists an open subset U of $\text{cap}_{\omega,m}$ less than ϵ such that φ_j, φ are continuous on $X \setminus U$ which is compact. By Dini's Theorem, φ_j converges uniformly to φ on $X \setminus U$. \square

We have obvious inclusions

$$SH_m(X, \omega) \cap \mathcal{C}^2(X) \subset \mathcal{P}_m(X, \omega) \subset SH_m(X, \omega),$$

and

$$PSH(X, \omega) \subset \mathcal{P}_m(X, \omega).$$

Remark 3.7. At this moment, we do not know if any function in $SH_m(X, \omega)$ can be approximated by decreasing sequence of \mathcal{C}^2 functions in the same class (even if the function in question is continuous). However, the class $\mathcal{P}_m(X, \omega)$ is sufficient for our purpose.

Remark 3.8. Quasi-uniform convergence implies convergence point wise outside a subset of $\text{Cap}_{\omega,m}$ zero. Moreover, if φ_j is uniformly bounded and converges quasi-uniformly to φ , then we have convergence in L^p for every $p > 0$. Indeed, Fix $\epsilon > 0$ and U an open subset as in definition 3.3, we have (by (3.1))

$$\begin{aligned} \int_X |\varphi_j - \varphi|^p \omega^n &\leq \int_{X \setminus U} |\varphi_j - \varphi|^p \omega^n + \int_U |\varphi_j - \varphi|^p \omega^n \\ &\leq \int_{X \setminus U} |\varphi_j - \varphi|^p \omega^n + \sup_{X,j} |\varphi_j - \varphi|^p \cdot \text{cap}_{\omega,m}(U) \\ &\leq \int_{X \setminus U} |\varphi_j - \varphi|^p \omega^n + C\epsilon. \end{aligned}$$

Taking the limsup over j and then letting $\epsilon \rightarrow 0$ we obtain

$$\limsup_j \|\varphi_j - \varphi\|_p = 0.$$

Lemma 3.9. If φ, ψ belong to the class $\mathcal{P}_m(X, \omega)$ then so does $\max(\varphi, \psi)$.

Proof. Let $(\varphi_j), (\psi_j)$ be uniformly bounded sequences of functions in $SH_m(X, \omega) \cap \mathcal{C}^2(X)$ converging quasi-uniformly to φ, ψ respectively. Set

$$u := \max(\varphi, \psi); \quad u_j := \max(\varphi_j, \psi_j); \quad v_j := \frac{1}{j} \log(e^{j\varphi_j} + e^{j\psi_j}).$$

For each $\epsilon > 0$ there exists an open subset U of $\text{cap}_{\omega,m}$ less than ϵ and φ_j, ψ_j converges uniformly on $X \setminus U$ to φ, ψ respectively. Since $u_j \leq v_j \leq \log(2)/j + u_j$ and u_j converges uniformly to u on $X \setminus U$ we deduce that v_j converges uniformly to u on $X \setminus U$. \square

3.2 Hessian measure

In this section we define complex Hessian measure for function in $SH_m(X, \omega)$ which can be approximated in $\text{cap}_{\omega, m}$ by \mathcal{C}^2 -functions in $SH_m(X, \omega)$. In particular, for functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$ this notion of Hessian measure can be defined by Bedford-Taylor method.

Theorem 3.10. *Let $\varphi \in SH_m(X, \omega)$ such that there exists a uniformly bounded sequence (φ_j) of \mathcal{C}^2 (ω, m) -subharmonic functions converging in $\text{cap}_{\omega, m}$ to φ . Then the sequence of measures*

$$H_m(\varphi_j) := (\omega + dd^c \varphi_j)^m \wedge \omega^{n-m}$$

converges (weakly in the sense of measures) to a positive Radon measure μ . Moreover, the measure μ does not depend on the choice of the approximating sequence (φ_j) . We define the Hessian measure of φ to be $H_m(\varphi) := \mu$.

Proof. Since all the measures $H_m(\varphi_j)$ have uniformly bounded mass (which is $\int_X \omega^n$), they stay in a weakly compact subset. It suffices to show that all accumulation points of this sequence are just the same. To do this it is enough to show that for every test function $\chi \in \mathcal{C}^\infty(X)$,

$$\lim_{j, k \rightarrow \infty} \int_X \chi [H_m(\varphi_j) - H_m(\varphi_k)] = 0.$$

By integration by part formula we have

$$\begin{aligned} (3.2) \quad \int_X \chi [H_m(\varphi_j) - H_m(\varphi_k)] &= \int_X \chi dd^c(\varphi_j - \varphi_k) \wedge T \\ &= \int_X (\varphi_j - \varphi_k) dd^c \chi \wedge T, \end{aligned}$$

where

$$T = \sum_{l=0}^{m-1} (\omega + dd^c \varphi_j)^l \wedge (\omega + dd^c \varphi_k)^{m-1-l} \wedge \omega^{n-m}.$$

Fix $\epsilon > 0$, and set $U = U(j, k, \epsilon) = \{|\varphi_j - \varphi_k| \geq \epsilon\}$. By C we will denote a constant that does not depend on j, k, ϵ . Then by (3.2) and (3.1) there exists $C > 0$ such that

$$\begin{aligned} \left| \int_X \chi [H_m(\varphi_j) - H_m(\varphi_k)] \right| &\leq \int_U |\varphi_j - \varphi_k| C \omega \wedge T + \int_{X \setminus U} |\varphi_j - \varphi_k| C \omega \wedge T \\ &\leq C \text{cap}_{\omega, m}(U) + C \epsilon \sup_{X \setminus U} |\varphi_j - \varphi_k| \int_{X \setminus U} \omega \wedge T. \end{aligned}$$

Now, it follows that

$$\limsup_{j,k \rightarrow \infty} \left| \int_X \chi [H_m(\varphi_j) - H_m(\varphi_k)] \right| \leq C\epsilon.$$

Let $\epsilon \rightarrow 0$ we obtain the result. For the independence in the choice of the sequence it is enough to repeat the above arguments. \square

For functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$ we can also define the Hessian measure in a weak sense following Bedford-Taylor method.

Lemma 3.11. *Let $\varphi_1, \varphi_2 \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$. Then the current $\omega_{\varphi_1} \wedge \omega_{\varphi_2} \wedge \omega^{n-m}$ is well defined in the weak sense (Bedford-Taylor), symmetric and (ω, m) -positive. Then we can define inductively the Hessian measure of $\varphi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$,*

$$H_m(\varphi) := (\omega + dd^c \varphi)^m \wedge \omega^{n-m}.$$

Moreover, this definition coincides with the one in Theorem 3.10.

Proof. It follows from definition of (ω, m) -subharmonic functions that $T_1 = (\omega + dd^c \varphi_1) \wedge \omega^{n-m}$ is a (ω, m) -positive current. If $\varphi_2 \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$ then $dd^c \varphi_2 \wedge T_1$ is the current defined by

$$dd^c \varphi_2 \wedge T_1 = dd^c(\varphi_2 T_1).$$

We denote by $T_2 = \omega_{\varphi_1} \wedge \omega_{\varphi_2} \wedge \omega^{n-m}$. Since φ_1, φ_2 are in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$, there exist uniformly bounded sequences $(\varphi_1^j), (\varphi_2^j)$ in $\mathcal{P}_m(X, \omega) \cap C^2(X)$ converging quasi-uniformly to φ_1, φ_2 respectively. The sequence of currents $T_2^j = \omega_{\varphi_1^j} \wedge \omega_{\varphi_2^j} \wedge \omega^{n-m}$ converges to T_2 and hence T_2 is (ω, m) -positive and

$$\omega_{\varphi_1} \wedge \omega_{\varphi_2} \wedge \omega^{n-m} = \omega_{\varphi_2} \wedge \omega_{\varphi_1} \wedge \omega^{n-m}.$$

To prove that T_2^j converges to T_2 , let us choose some test form χ and prove the following convergence

$$(3.3) \quad \lim_{j \rightarrow \infty} \int_X \chi \wedge dd^c(\varphi_2^j - \varphi_2) \wedge T_1 = 0.$$

We have

$$\begin{aligned} \left| \int_X \chi \wedge dd^c(\varphi_2^j - \varphi_2) \wedge T_1 \right| &= \left| \int_X (\varphi_2^j - \varphi_2) dd^c \chi \wedge T_1 \right| \\ &\leq C \int_X |\varphi_2^j - \varphi_2| \omega_{\varphi_1} \wedge \omega^{n-1}, \end{aligned}$$

where the constant C depends only on χ, ω . Now (3.3) follows from the last inequality in view of

$$\int_U \omega_{\varphi_1} \wedge \omega^{n-1} \leq C \text{cap}_{\omega, m}(U),$$

for every open subset $U \subset X$. \square

We can prove inductively that the current

$$T_k = \omega_{\varphi_1} \wedge \dots \wedge \omega_{\varphi_k} \wedge \omega^{n-m}$$

is well-defined, symmetric, (ω, m) -positive, for each $k \leq m$ and $\varphi_i \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$. The Hessian measure of $\varphi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$ is defined in this way $H_m(\varphi) = \omega_\varphi \wedge \dots \wedge \omega_\varphi \wedge \omega^{n-m}$. Now, given $\varphi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$, it is easy to see that the Hessian measure of φ defined by the above construction coincides with the Hessian measure $H_m(\varphi)$ defined in Theorem 3.10.

3.3 Some Convergence results

In this section we prove some convergence results and the comparison principle for functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$.

Proposition 3.12. *Let $(\varphi_j^1), \dots, (\varphi_j^m)$ be uniformly bounded sequence of functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$ converging quasi-uniformly to $\varphi^1, \dots, \varphi^m$ respectively. Then we have the weak convergence of measures*

$$\omega_{\varphi_j^1} \wedge \dots \wedge \omega_{\varphi_j^m} \wedge \omega^{n-m} \rightharpoonup \omega_{\varphi^1} \wedge \dots \wedge \omega_{\varphi^m} \wedge \omega^{n-m}.$$

Moreover, if $u \in \mathcal{P}_m(X, \omega)$, we also have

$$(3.4) \quad u\omega_{\varphi_j^1} \wedge \dots \wedge \omega_{\varphi_j^m} \wedge \omega^{n-m} \rightharpoonup u\omega_{\varphi^1} \wedge \dots \wedge \omega_{\varphi^m} \wedge \omega^{n-m}.$$

In particular $uH_m(\varphi_j^1) \rightharpoonup uH_m(\varphi^1)$.

Proof. Using the identity

$$\begin{aligned} & \omega_{v_1} \wedge \dots \wedge \omega_{v_m} \wedge \omega^{n-m} - \omega_{u_1} \wedge \dots \wedge \omega_{u_m} \wedge \omega^{n-m} \\ &= \sum_j \omega_{u_1} \wedge \omega_{u_2} \wedge \dots \wedge \omega_{u_{j-1}} \wedge dd^c(v_j - u_j) \wedge \omega_{v_{j+1}} \wedge \dots \wedge \omega_{v_m} \wedge \omega^{n-m}. \end{aligned}$$

we reduce the proof to showing that if $v_j \rightarrow v$ quasi-uniformly with v_j uniformly bounded and closed (ω, m) -positive currents T_j have the form $\omega_{v_j^1} \wedge \dots \wedge \omega_{v_j^{m-1}} \wedge \omega^{n-m}$ with $v_j^s \in \mathcal{P}_m(X, \omega), 0 \leq v_j^s \leq 1$ then

$$(3.5) \quad dd^c(v_j - v) \wedge T_j \rightarrow 0.$$

Fix a test function χ . For $\epsilon > 0$ let U be an open subset of $\text{cap}_{\omega, m}$ less than ϵ and v_j converges uniformly to v on $X \setminus U$. We have

$$\begin{aligned} & \left| \int_X \chi dd^c(v_j - v) \wedge T_j \right| = \left| \int_X (v_j - v) dd^c \chi \wedge T_j \right| \\ & \leq \left| \int_U (v_j - v) dd^c \chi \wedge T_j \right| + \left| \int_{X \setminus U} (v_j - v) dd^c \chi \wedge T_j \right| \\ & \leq C \left(\int_U \omega \wedge T_j + \int_{X \setminus U} |v_j - v| \omega \wedge T_j \right) \\ & \leq C \cdot \text{cap}_{\omega, m}(U) + C \cdot \sup_{X \setminus U} |v_j - v|, \end{aligned}$$

where C is a constant depending on the test function χ and the uniform bound for $|v_j - v|$. Now let $j \rightarrow \infty$ and then $\epsilon \rightarrow 0$ we obtain (3.5) and hence (3.13). To prove (3.4), set

$$T_j = \omega_{\varphi_j^1} \wedge \dots \wedge \omega_{\varphi_j^m} \wedge \omega^{n-m},$$

$$T = \omega_{\varphi^1} \wedge \dots \wedge \omega_{\varphi^m} \wedge \omega^{n-m}.$$

Fix a test function χ and a small number $\epsilon > 0$. There exists an open subset $U \subset X$ of $\text{cap}_{\omega, m}$ less than ϵ and a C^2 approximating sequence (u_j) of u such that (u_j) converges uniformly to u in $K = X \setminus U$. For each $j, k \in \mathbb{N}$ we have

$$\begin{aligned} \left| \int_X \chi(uT_j - uT) \right| & \leq \left| \int_X \chi(u - u_k)(T_j - T) \right| + \left| \int_X \chi u_k(T_j - T) \right| \\ & \leq C(\text{Cap}_{\omega, m}(U) + \|u_k - u\|_{L^\infty(K)}) + \left| \int_X \chi u_k(T_j - T) \right|, \end{aligned}$$

where C is a positive constant depending only on $\sup_k \|u_k - u\|_{L^\infty(X)}$ and $\|\chi\|_{L^\infty(X)}$. Now let j tend to ∞ and then $k \rightarrow \infty$, we obtain

$$\limsup_j \left| \int_X \chi(uT_j - uT) \right| \leq C\epsilon,$$

and since ϵ can be chosen arbitrarily small, we obtain (3.4). \square

Corollary 3.13. *Let $(\varphi_j^0), \dots, (\varphi_j^m)$ be uniformly bounded sequence of functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$ converging quasi-uniformly to $\varphi^0, \dots, \varphi^m$ respectively. Then we have the weak convergence of measures*

$$\varphi_j^0 \omega_{\varphi_j^1} \wedge \dots \wedge \omega_{\varphi_j^m} \wedge \omega^{n-m} \rightharpoonup \varphi^0 \omega_{\varphi^1} \wedge \dots \wedge \omega_{\varphi^m} \wedge \omega^{n-m}.$$

Proof. Let χ be a test function. By Proposition 3.12 it suffices to show that

$$\int_X \chi(\varphi_j^0 - \varphi^0) \omega_{\varphi_j^1} \wedge \dots \wedge \omega_{\varphi_j^m} \wedge \omega^{n-m} \longrightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Now, we just apply the definition of quasi-uniform convergence as we have done many times before. \square

The integration by parts formula is valid for \mathcal{C}^2 functions (by Stokes). By corollary 3.13 we see that it is also valid for functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$.

Theorem 3.14 (Integration by parts). *Let $\varphi, \psi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$ and T be a current of the form*

$$T = \omega_{\varphi_1} \wedge \dots \wedge \omega_{\varphi_{m-1}} \wedge \omega^{n-m},$$

with $\varphi_i \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$. Then

$$\int_X \varphi dd^c \psi \wedge T = \int_X \psi dd^c \varphi \wedge T.$$

Theorem 3.15 (Maximum principle). *If φ, ψ be two functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$ then*

$$\mathbf{1}_{\{\varphi > \psi\}} H_m(\max(\varphi, \psi)) = \mathbf{1}_{\{\varphi > \psi\}} H_m(\varphi).$$

Proof. Let $(\varphi_j), (\psi_j)$ be \mathcal{C}^2 approximating sequences of φ and ψ respectively. It is obvious that

$$\mathbf{1}_{\{\varphi_j > \psi_j\}} H_m(\max(\varphi_j, \psi_j)) = \mathbf{1}_{\{\varphi_j > \psi_j\}} H_m(\varphi_j).$$

For $u_j = \max(\varphi_j - \psi_j, 0) = \max(\varphi_j, \psi_j) - \psi_j$, we have $u_j \cdot \mathbf{1}_{\{\varphi_j > \psi_j\}} = u_j$. Thus,

$$u_j H_m(\max(\varphi_j, \psi_j)) = u_j H_m(\varphi_j).$$

Since u_j is a difference of two functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$, we can apply corollary 3.13 to obtain

$$u H_m(\max(\varphi, \psi)) = u H_m(\varphi),$$

where $u = \max(\varphi, \psi) - \psi$. Now, for each $\epsilon > 0$ we have

$$\frac{u}{u + \epsilon} H_m(\max(\varphi, \psi)) = \frac{u}{u + \epsilon} H_m(\varphi).$$

Thus, the monotone convergence (as $\epsilon \rightarrow 0$) yields

$$\mathbf{1}_{\{\varphi > \psi\}} H_m(\max(\varphi, \psi)) = \mathbf{1}_{\{\varphi > \psi\}} H_m(\varphi).$$

□

From Theorem 3.15 we easily get

Corollary 3.16 (Comparison principle). *If $\varphi, \psi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$ then*

$$\int_{(\varphi > \psi)} H_m(\varphi) \leq \int_{(\varphi > \psi)} H_m(\psi).$$

Lemma 3.17. *Let φ, ψ be two non positive functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$. If $s > 0$ and $0 < t < 1$ then we have*

$$t^m \text{Cap}_{\omega, m}(\varphi - \psi < -t - s) \leq (1 + M)^m \int_{(\varphi - \psi < -s)} H_m(\varphi),$$

where $M = \|\psi\|_{L^\infty(X)}$.

Proof. We can assume that ψ is continuous on X . For the general case we can approximate ψ quasi-uniformly by sequence of \mathcal{C}^2 functions in $SH_m(X, \omega)$.

Let u be a function in $\mathcal{P}_m(X, \omega) \cap \mathcal{C}^2(X)$ such that $0 \leq u \leq 1$. Set $\delta = \frac{t}{M+1}$, $\rho = \delta u + (1 - \delta)\psi - t - s$. Observe that

$$\{\varphi - \psi < -t - s\} \subset \{\varphi < \rho\} \subset \{\varphi - \psi < -s\}.$$

By the comparison principle we have

$$\begin{aligned} \delta^m \int_{(\varphi - \psi < -s - t)} H_m(u) &= \int_{(\varphi - \psi < -s - t)} (\delta \omega_u)^m \wedge \omega^{n-m} \\ &\leq \int_{(\varphi - \psi < -s - t)} (\delta \omega_u + (1 - \delta) \omega_\psi)^m \wedge \omega^{n-m} \\ &\leq \int_{(\varphi < \rho)} \omega_\rho^m \wedge \omega^{n-m} \\ &\leq \int_{(\varphi < \rho)} \omega_\varphi^m \wedge \omega^{n-m} \\ &\leq \int_{(\varphi - \psi < -s)} \omega_\varphi^m \wedge \omega^{n-m}. \end{aligned}$$

Thus,

$$\delta^m \int_{(\varphi-\psi < -s-t)} H_m(u) \leq \int_{(\varphi-\psi < -s)} \omega_\varphi^m \wedge \omega^{n-m}.$$

Now, it suffices to take the supremum over all $u \in \mathcal{P}_m(X, \omega) \cap \mathcal{C}^2(X)$. \square

Proposition 3.18 (Chern-Levine-Nirenberg inequality). *Let T be any current of the form $T = \omega_{u_1} \wedge \dots \wedge \omega_{u_{m-1}} \wedge \omega^{n-m}$ with $u_1, \dots, u_{m-1} \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$, and φ, ψ be two functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$. Then*

$$(3.6) \quad \int_X |\psi| \omega_\varphi \wedge T \leq \int_X |\psi| T \wedge \omega + \left(2|\sup_X \psi| + \sup_X \varphi - \inf_X \varphi \right) \int_X \omega^n.$$

Proof. The proof is nearly the same as in [23]. We first assume that $\psi \leq 0 \leq \varphi$. Integrating by part (Theorem 3.14) allows us to do this) we obtain

$$\int_X (-\psi) \omega_\varphi \wedge T = \int_X (-\psi) \omega \wedge T + \int_X \varphi (-dd^c \psi) \wedge T.$$

Since $\varphi \geq 0$ and $(\omega + dd^c \psi) \wedge T \geq 0$, it follows that

$$\int_X \varphi (-dd^c \psi) \wedge T \leq \int_X \varphi \omega \wedge T \leq (\sup_X \varphi) \int_X \omega^n.$$

Thus we have

$$(3.7) \quad \int_X (-\psi) \omega_\varphi \wedge T \leq \int_X (-\psi) T \wedge \omega + (\sup_X \varphi) \int_X \omega^n.$$

For general functions $\varphi, \psi \in \mathcal{P}_m(X, \omega)$ we apply (3.7) for $\tilde{\varphi} = \varphi - \inf_X \varphi$ and $\tilde{\psi} = \psi - \sup_X \psi$ to obtain

$$\begin{aligned} \int_X |\psi| \omega_\varphi \wedge T &\leq \int_X |\tilde{\psi}| \omega_\varphi \wedge T + |\sup_X \psi| \int_X \omega^n \\ &= \int_X |\tilde{\psi}| \omega_{\tilde{\varphi}} \wedge T + |\sup_X \psi| \int_X \omega^n \\ (\text{apply (3.7)}) &\leq \int_X |\tilde{\psi}| T \wedge \omega + (|\sup_X \psi| + \sup_X \tilde{\varphi}) \int_X \omega^n \\ &\leq \int_X |\psi| T \wedge \omega + (2|\sup_X \psi| + \sup_X \varphi - \inf_X \varphi) \int_X \omega^n. \end{aligned}$$

\square

Applying (3.6) for $T_i = \omega_\varphi^i \wedge \omega^{n-m+i}$ for $i = m-1, \dots, 0$ we obtain

Corollary 3.19. *Let φ, ψ be two functions in $\mathcal{P}_m(X, \omega)$ such that $0 \leq \varphi \leq 1$. Then*

$$\int_X |\psi| H_m(\varphi) \leq \int_X |\psi| \omega^n + m \left(2 \sup_X \psi + 1 \right) \int_X \omega^n.$$

Corollary 3.20. *There exists a constant $C > 0$ such that for all $\psi \in \mathcal{P}_m(X, \omega)$ such that $\sup_X \psi = -1$ and $t > 0$ then*

$$\text{Cap}_{\omega, m}(\psi < -t) \leq C/t.$$

4 Stability of solution

In this section we use a Moser iterative process to establish a uniform a priori estimate on the solutions of complex Hessian equation and stability property. By Green formula we easily get

Proposition 4.1. *Let (X, ω) be a compact Kähler manifold of complex dimension n and q be a positive number such that $q < \frac{n}{n-1}$. Then for every smooth ω -subharmonic function φ such that*

$$\int_X \varphi \omega^n = 0$$

we have the following estimate

$$\|\varphi\|_q \leq C,$$

where C is a constant depending on X, q .

The following uniform estimate has been proved in [28] with the constant depending on uniform norm of f . We slightly modify the proof so that the constant depends on L^p norm of f , for $p > n$ instead of $\|f\|_{L^\infty(X)}$.

Theorem 4.2. *Let φ be a smooth normalized solution to*

$$(\omega + dd^c \varphi)^m \wedge \omega^{n-m} = f \omega^n$$

with $f > 0$ smooth. Let $q > n$ be a real number. Then we have the following uniform estimate

$$\|\varphi\|_{L^\infty} \leq C,$$

where the constant C depends on X, q and the upper bound for $\|f - 1\|_{L^q}$.

Proof. Without loss of generality we can assume that $\int_X \omega^n = 1$. We have

$$\int_X (1-f)\varphi|\varphi|^{p-2}\omega^n = \int_X |\varphi|^{p-2}\varphi dd^c(-\varphi) \wedge T,$$

with

$$T = \sum_{k=0}^{m-1} \omega_\varphi^k \wedge \omega^{n-k-1} \geq \omega^{n-1}.$$

A simple application of Stokes formula yields

$$\int_X |\varphi|^{p-2}\varphi dd^c(-\varphi) \wedge \omega^{n-1} = (p-1) \int_X |\varphi|^{p-2} d\varphi \wedge d^c\varphi \wedge \omega^{n-1}.$$

It together with $|\nabla|\varphi|^{p/2}|^2 = \frac{p^2}{4}|\varphi|^{p-2}|\nabla\varphi|^2$ give us the following inequality

$$\begin{aligned} (4.1) \quad \int_X |\nabla|\varphi - \psi|^{p/2}|^2 \omega^n &\leq \frac{np^2}{2(p-1)} \int_X (1-f)\varphi|\varphi|^{p-2}\omega^n \\ &\leq \frac{np^2}{2(p-1)} \|1-f\|_q \|\varphi\|_{(p-1)q'}^{p-1}. \end{aligned}$$

By Sobolev inequality we have

$$(4.2) \quad \|\varphi\|^p_{\frac{n}{n-1}} = \|\varphi\|_{\frac{2n}{n-1}}^2 \leq C \left(\int_X |\nabla|\varphi|^{p/2}|^2 \omega^n + \int_X |\varphi|^p \omega^n \right).$$

Recall that $\int_X \omega^n = 1$. For simplicity we denote $\|f-1\|_q = C_q$. Now by (4.1) and (4.2) we obtain

$$\begin{aligned} (4.3) \quad \|\varphi\|^p_{\frac{n}{n-1}} &\leq Cp \left(C_q \|\varphi\|_{(p-1)q'}^{p-1} + \|\varphi\|_p^p \right) \\ &\leq Cp \left(C_q \|\varphi\|_{pq'}^{p-1} + \|\varphi\|_{pq'}^p \right). \end{aligned}$$

We want to bound $\|\varphi\|_{L^\infty}$. So we can assume that all the L^p norms of φ are greater than 1. We obtain from (4.3) that

$$\|\varphi\|^p_{\frac{n}{n-1}} \leq Cp(C_q + 1) \|\varphi\|_{pq'}^p,$$

and hence

$$(4.4) \quad \|\varphi\|_{p\frac{n}{n-1}} \leq (Cp)^{1/p} \|\varphi\|_{pq'}.$$

By Sobolev-Poincare inequality we have

$$(4.5) \quad \|\varphi\|_2 \leq C \cdot \|\nabla\varphi\|_2 = C \cdot \|\nabla|\varphi|\|_2,$$

since $\nabla\varphi = \nabla|\varphi|$ almost everywhere. From (4.1) for $p = 2$ and (4.5) and Proposition 4.1 we obtain

$$(4.6) \quad \|\varphi\|_{\frac{2n}{n-1}} \leq C.C_q.\|\varphi\|_{q'} \leq C.C_q.$$

To simplify the notation we use the following quantities

$$a := \frac{n}{(n-1)q'} > 1; \quad \beta_k := 2a^k, k = 0, 1, \dots \quad \gamma_k := 1/\beta_k.$$

Note that $\beta_k > 1$ for every $k \geq 0$ and since $\beta_k = a\beta_{k-1}$ with $a > 1$ the product $\prod_{j=1}^k \beta_j^{\gamma_j}$ is convergent to $\beta > 1$. Now by inequality (4.4) applied for $p = 2a^k, k = 1, 2, 3, \dots$ we have

$$\begin{aligned} \|\varphi\|_{\frac{2n}{n-1}.a^k} &\leq (C\beta_k(C_q + 1))^{\gamma_k} \|\varphi\|_{\frac{2n}{n-1}.a^{k-1}} \\ &\leq (C\beta_k(C_q + 1))^{\gamma_k} . (C\beta_{k-1}(C_q + 1))^{\gamma_{k-1}} . \|\varphi\|_{\frac{2n}{n-1}.a^{k-2}} \\ &\leq \dots \end{aligned}$$

Therefore,

$$(4.7) \quad \|\varphi\|_{L^\infty(X)} \leq (C.(C_q + 1))^\gamma . \beta . \|\varphi\|_{\frac{2n}{n-1}}.$$

with

$$\beta = \prod_{k=1}^{\infty} (\beta_k)^{\gamma_k}, \quad \gamma = \sum_{k=1}^{\infty} \gamma_k.$$

Now, from (4.6) and (4.7) we have

$$\|\varphi\|_{L^\infty(X)} \leq C.$$

□

Theorem 4.3. *Let $\varphi, \psi \in SH_m(X, \omega) \cap \mathcal{C}^2(X, \omega)$, $r \geq 2$, and set $\rho = \varphi - \psi$. Then*

$$\int_X |\rho|^{r-2} d\rho \wedge d^c \rho \wedge \omega^{n-1} \leq C \left(\int_X |\rho|^{r-2} \rho (H_m(\psi) - H_m(\varphi)) \right)^{2^{1-m}},$$

where C is a positive constant depending only on n, m, r , and upper bounds of $\|\varphi\|_{L^\infty(X)}$, $\|\psi\|_{L^\infty(X)}$, and $\int_X \omega^n$.

Proof. We follow the idea of Blocki [3]. We prove inductively for $k = 0, 1, \dots, m-1$ that

$$\int_X |\rho|^{r-2} d\rho \wedge d^c \rho \wedge \omega_\varphi^j \wedge \omega_\psi^{m-1-k-j} \wedge \omega^{n-m+k} \leq C.b^{2^{-k}},$$

for every $j = 0, \dots, m-1-k$, where

$$b := \int_X |\rho|^{r-2} \rho (H_m(\psi) - H_m(\varphi)).$$

For $k = 0$, it is trivially true. Suppose that it is true for $0, 1, \dots, k-1$. We have

$$\omega_\varphi^j \wedge \omega_\psi^{m-1-k-j} \wedge \omega^{n-m+k} = \omega_\varphi^{j+k} \wedge \omega_\psi^{m-1-k-j} \wedge \omega^{n-m} - dd^c \varphi \wedge \alpha,$$

with

$$\alpha = \omega_\varphi^j \wedge \omega_\psi^{m-1-k-j} \wedge \omega^{n-m} \wedge \sum_{l=0}^{k-1} \omega_\varphi^l \wedge \omega^{k-1-l}.$$

Moreover,

$$\begin{aligned} (4.8) \quad & |\rho|^{r-2} d\rho \wedge d^c \rho \wedge dd^c \varphi \wedge \alpha = \frac{1}{r-1} d\chi(\rho) \wedge d^c \rho \wedge dd^c \varphi \wedge \alpha \\ & = \frac{1}{p-1} \left[d \left(d\chi(\rho) \wedge d^c \rho \wedge d^c \varphi \wedge \alpha \right) + d\chi(\rho) \wedge dd^c \rho \wedge d^c \varphi \wedge \alpha \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} (4.9) \quad & \int_X |\rho|^{r-2} d\rho \wedge d^c \rho \wedge \omega_\varphi^j \wedge \omega_\psi^{m-1-k-j} \wedge \omega^{n-m+k} \\ & \leq \int_X |\rho|^{r-2} d\rho \wedge d^c \rho \wedge T - \frac{1}{r-1} \int_X d\chi(\rho) \wedge dd^c \rho \wedge d^c \varphi \wedge \alpha. \end{aligned}$$

Since $\int_X |\rho|^{r-2} d\rho \wedge d^c \rho \wedge T = b \leq C$, it remains to estimate the last term in the right hand side of (4.9). We have

$$- \int_X d\chi(\rho) \wedge dd^c \rho \wedge d^c \varphi \wedge \alpha \leq \left| \int_X d\chi(\rho) \wedge d^c \varphi \wedge \alpha \wedge \omega_\varphi \right| + \left| \int_X d\chi(\rho) \wedge d^c \varphi \wedge \alpha \wedge \omega_\psi \right|.$$

For η we denote φ or ψ . Cauchy-Schwartz inequality yields

$$\begin{aligned} & \left| \int_X d\chi(\rho) \wedge d^c \varphi \wedge \alpha \wedge \omega_\eta \right| \\ & \leq \left(\int_X d\chi(\rho) \wedge d^c \chi(\rho) \wedge \alpha \wedge \omega_\eta \right)^{1/2} \left(\int_X d\varphi \wedge d^c \varphi \wedge \alpha \wedge \omega_\eta \right)^{1/2}. \end{aligned}$$

By the induction hypothesis it remains to prove that

$$\int_X d\varphi \wedge d^c \varphi \wedge \alpha \wedge \omega_\eta \leq C.$$

But

$$\begin{aligned} \int_X d\varphi \wedge d^c \varphi \wedge \alpha \wedge \omega_\eta &= - \int_X \varphi \wedge dd^c \varphi \wedge \alpha \wedge \omega_\eta \\ &\leq \left| \int_X \varphi \wedge \omega_\varphi \wedge \alpha \wedge \omega_\eta \right| + \left| \int_X \varphi \omega \wedge \alpha \wedge \omega_\eta \right| \\ &\leq 2k \|\varphi\|_{L^\infty(X)}. \end{aligned}$$

Therefore, Theorem 4.3 is proved. \square

From Theorem 4.3 and Corollary 3.2 we easily get

Corollary 4.4. *Let $\varphi, \psi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$, and set $\rho = \varphi - \psi$. Then*

$$\int_X d\rho \wedge d^c \rho \wedge \omega^{n-1} \leq C \left(\int_X \rho (H_m(\psi) - H_m(\varphi)) \right)^{2^{1-m}},$$

where C is a positive constant depending only on n, m , and upper bounds of $\|\varphi\|_{L^\infty(X)}$, $\|\psi\|_{L^\infty(X)}$, and $\int_X \omega^n$.

From Theorem 4.2, Theorem 4.2, and again the Moser's iteration method we prove the following important stability theorem.

Theorem 4.5. *Let us consider two smooth strictly positive functions f, h on X and φ, ψ corresponding solutions to*

$$\omega_\varphi^m \wedge \omega^{n-m} = f\omega^n, \quad \omega_\psi^m \wedge \omega^{n-m} = h\omega^n$$

normalized by

$$\int_X \varphi \omega^n = \int_X \psi \omega^n = 0.$$

Given positive numbers $q > n$, $p \geq 1$ there exist positive constants C depending on $X, n, p, q, \|f\|_q, \|h\|_q$, and γ depending on n, p, q such that

$$\|\varphi - \psi\|_p \leq C \|f - h\|_q^\gamma.$$

Proof. For simplicity we denote $\|h - f\|_q$ by C_q and by C we denote a constant depending only on X, q, p, A . Let $r \geq 2$ and consider the function

$\chi : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\chi(t) = t|t|^{r-2}$. Then χ is a \mathcal{C}^1 function in \mathbb{R} with $\chi'(t) = (r-1) \cdot |t|^{r-2}$. Denote $\rho := \varphi - \psi$. By hypothesis we have

$$(\omega + dd^c \varphi)^m \wedge \omega^n = f\omega^n, \quad (\omega + dd^c \psi)^m \wedge \omega^{n-m} = h\omega^n.$$

Thus

$$\int_X \chi(\rho)(h-f)\omega^n = \int_X \chi(\rho)dd^c(\rho) \wedge T,$$

with

$$T = \sum_{k=0}^{m-1} \omega_\varphi^k \wedge \omega_\psi^{m-1-k} \wedge \omega^{n-m}.$$

From Theorem 4.3 we have

$$(4.10) \quad \int_X |\rho|^{r-2} d\rho \wedge d^c \rho \wedge \omega^{n-1} \leq C \cdot b^{2^{-m+1}},$$

with $b := \int_X |\rho|^{r-2} d\rho \wedge d^c \rho \wedge T$.

Now we use the Moser's Iteration process to establish the upper bound for $\|\varphi - \psi\|_{L^p(X)}$ in terms of $\|f - h\|_q$. Note that

$$|\nabla |\rho|^{r/2}|^2 = \frac{r^2}{4} |\rho|^{r-2} |\nabla \rho|^2.$$

Set $d = 2^{1-m}$. Using (4.10) we have the following estimate:

$$\begin{aligned} (4.11) \quad \int_X |\nabla |\rho|^{r/2}|^2 \omega^n &= \frac{r^2}{4} \int_X |\rho|^{r-2} |\nabla \rho|^2 \omega^n \\ &= \frac{nr^2}{2} \int_X |\rho|^{r-2} d\rho \wedge d^c \rho \wedge \omega^{n-1} \\ &\leq \frac{nr^2}{2} C \left(\int_X |\rho|^{r-2} d\rho \wedge d^c \rho \wedge T \right)^d \\ &\leq \frac{nr^2}{2(r-1)} C \left(\int_X |\rho|^{r-2} \rho \cdot (h-f) \omega^n \right)^d \\ &\leq Cr \|h-f\|_q^d \|\rho\|_{(r-1)q'}^{(r-1)d}. \end{aligned}$$

By Sobolev inequality we have

$$\begin{aligned} (4.12) \quad \|\rho\|^p_{\frac{n}{n-1}} &= \|\rho|^{r/2}\|_{\frac{2n}{n-1}}^2 \\ &\leq C \left(\int_X |\nabla |\rho|^{r/2}|^2 \omega^n + \int_X |\rho|^r \omega^n \right). \end{aligned}$$

Now by (4.11) and (4.12) we obtain

$$(4.13) \quad \begin{aligned} \|\rho\|^r \Big\|_{\frac{n}{n-1}} &\leq Cr \left(C_q^d \|\rho\|_{(r-1)q'}^{(r-1)d} + \|\rho\|_r^r \right) \\ &\leq Cr \left(C_q^d \|\rho\|_{rq'}^{(r-1)d} + \|\rho\|_{rq'}^r \right). \end{aligned}$$

By Sobolev-Poincare inequality we have

$$(4.14) \quad \|\rho\|_2 \leq C \cdot \|\nabla \rho\|_2 = C \cdot \|\nabla |\rho|\|_2,$$

since $\nabla \rho = \nabla |\rho|$ almost everywhere. From (4.11) for $r = 2$ and (4.14) and Proposition 4.1 we obtain

$$(4.15) \quad \|\rho\|_2 \leq C \cdot C_q^d \cdot \|\rho\|_{q'} \leq C \cdot C_q^d.$$

Thus, we can assume that $\|\rho\|_2 \leq 1$. With out loss of generality we can assume that all L^r norms ($r \geq 2$) of ρ are less than 1. By (4.13) we have

$$(4.16) \quad \|\rho\|_{r \frac{n}{n-1}} \leq C \cdot r^{1/r} \|\rho\|_{rq'}^{(1-1/r)d}.$$

To simplify the notation we use the following quantities

$$a := \frac{n}{(n-1)q'} > 1; \quad \beta_k := \frac{2a^k}{q'}, k = 0, 1, \dots \quad \gamma_k := 1/\beta_k.$$

Note that $\beta_k > 1$ for every $k \geq 0$ and since $\beta_k = a\beta_{k-1}$ with $a > 1$ the product $\prod_{j=1}^k \beta_j^{\gamma_j}$ is convergent to $\beta > 1$. The sequence $\{\prod_{j=1}^k (1 - \gamma_j)\}_k$ is also convergent and the limit γ is between $e^{\frac{-C_1}{a-1}}$ and $e^{\frac{-C_2}{a-1}}$ with two positive constant C_1, C_2 . Now by inequality (4.16) applied for $r = 2a^k, k = 1, 2, \dots$ we have

$$\begin{aligned} \|\rho\|_{2a^{k+1}} &\leq (C\beta_k)^{\gamma_k} \|\rho\|_{2a^k}^{d(1-\gamma_k)} \\ &\leq (C\beta_k)^{\gamma_k} \cdot (C\beta_{k-1})^{d(1-\gamma_k) \cdot \gamma_{k-1}} \cdot \|\rho\|_{2a^{k-1}}^{(1-\gamma_k)(1-\gamma_{k-1})} \\ &\leq (C\beta_k)^{\gamma_k} \cdot (C\beta_{k-1})^{\gamma_{k-1}} \cdot \|\rho\|_{2a^{k-1}}^{d^2(1-\gamma_k)(1-\gamma_{k-1})} \\ &\leq \dots \end{aligned}$$

Therefore,

$$(4.17) \quad \|\varphi - \psi\|_p \leq C \cdot \beta \cdot \|\varphi - \psi\|_2^{d^k \gamma}.$$

with

$$\beta = \prod_{k=0}^{\infty} \beta_k^{1/\beta_k}, \quad \gamma = \prod_{k=0}^{\infty} (1 - \gamma_k), \quad 2a^k \geq p.$$

Now, from (4.15) and (4.17) we have

$$\|\varphi - \psi\|_{L^p(X)} \leq C \cdot \|f - h\|_q^\gamma,$$

which completes the proof of Theorem 4.5. \square

Theorem 4.6. *Let (X, ω) be a n -dimensional compact Kähler manifold of non negative holomorphic bisectional curvature. Fix $1 \leq m \leq n < p$ and let $0 \leq f \in L^p(X)$ such that $\int_X f \omega^n = \int_X \omega^n$. Then equation (1.1) has a bounded weak solution $\varphi \in SH_m(X, \omega)$.*

Proof. Let (f_j) be a sequence of smooth strictly positive functions on X converging in $L^p(X)$ to f . We can assume that $\int_X f_j \omega^n = 1, \forall j$. From [?] there exists, for each j , a smooth (ω, m) -subharmonic solution φ_j , normalized by $\sup_X \varphi_j = 0$. By Theorem (4.5) and Theorem (4.2) we have

$$\sup_j \|\varphi_j\|_{L^\infty(X)} < +\infty$$

$$(4.18) \quad \lim_{j,k \rightarrow +\infty} \|\varphi_j - \varphi_k\|_{L^q(X)} = 0, \quad \forall q > 0.$$

We claim that $\lim_{j,k \rightarrow \infty} \text{cap}_{\omega,m}(|\varphi_j - \varphi_k| > 2\epsilon) = 0$ for every small $\epsilon > 0$. Applying Lemma 3.17 with $t = s = \epsilon$ we obtain

$$\begin{aligned} \text{cap}_{\omega,m}(\varphi_j - \varphi_k < -2\epsilon) &\leq (1+M)^m \epsilon^{-1} \int_{(\varphi_j - \varphi_k < -\epsilon)} H_m(\varphi_j) \\ &\leq \frac{(1+M)^m}{\epsilon^{2m}} \int_X |\varphi_j - \varphi_k| f_j \omega^n \\ &\leq \frac{(1+M)^m}{\epsilon^{2m}} \|\varphi_j - \varphi_k\|_{L^{p'}(X)} \cdot \|f_j\|_{L^p(X)}. \end{aligned}$$

By interchanging j and k and noting that $\|f_j\|_{L^p(X)}$ is uniformly bounded and using (4.18) we obtain

$$(4.19) \quad \lim_{j,k \rightarrow \infty} \text{cap}_{\omega,m}(|\varphi_j - \varphi_k| > 2\epsilon) = 0.$$

From (4.19), we can find a subsequence of (φ_j) , still denoted by (φ_j) , such that

$$\text{cap}_{\omega,m}\left(|\varphi_j - \varphi_{j+1}| > \frac{1}{j^2}\right) \leq 2^{-j}.$$

Set $\varphi = (\limsup_{j \rightarrow \infty} \varphi_j)^*$. It is easy to see that $\varphi \in SH_m(X, \omega)$ and $\varphi_j \rightarrow \varphi$ in $\text{cap}_{\omega, m}$. Thus the Hessian measure of φ is well-defined thanks to Theorem 3.10 and hence φ is a bounded weak solution of

$$(\omega + dd^c \varphi)^m \wedge \omega^{n-m} = f \omega^n.$$

□

Remark 4.7. Using Moser iteration method, it is not clear that we can prove the continuity of the solution. In the case of Monge-Ampère equation, Kolodziej [32] introduced a very powerful method which has been adapted also to the complex Hessian equation by Kolodziej and Dinew (see [17]). We summarize it in the next section.

5 Continuity of the solution

Continuity of our solution follows from a very recent work of Dinew and Kolodziej [17] which we learned about when preparing this paper. The main idea is the comparison estimate between volume and capacity. Here we will use this idea and follow the lines in [18] to prove a weak $L^\infty - L^1$ stability which implies the continuity of the solution.

Definition 5.1. Let $\alpha > 0, A > 0$. A Borel measure μ on X satisfies condition $\mathcal{Q}_m(\alpha, A, \omega)$ if for all Borel subsets K of X ,

$$\mu(K) \leq A \text{Cap}_{\omega, m}(K)^{1+\alpha}.$$

Proposition 5.2. *Let μ be a Borel measure satisfying condition $\mathcal{Q}_m(\alpha, A, \omega)$. Suppose that $\varphi \in \mathcal{P}_m(X, \omega)$ solves $H_m(\varphi) = \mu$, and $\sup_X \varphi = -1$. Then there exists a constant $C = C(\alpha, A, \omega, n, m)$ such that*

$$\sup_X |\varphi| \leq C.$$

Sketch of proof. Set

$$f(s) := [\text{Cap}_{\omega, m}(\varphi < -s)]^{1/m}.$$

Observe that $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is right continuous, decreasing with $\lim_{s \rightarrow \infty} f = 0$. Since μ satisfies condition $\mathcal{Q}_m(\alpha, A, \omega)$, it follows from Lemma 3.17 applied to the function $\psi \equiv 0$ that f satisfies the condition in Lemma 2.4 in [18]. Moreover it follows from Corollary 3.20 that

$$f(s) \leq C_1 s^{-1/m},$$

for some constant C_1 which only depends on ω . Thus, follow the lines in [18], page 615, we have the desired uniform estimate.

Theorem 5.3. *Suppose that $\varphi, \psi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$ satisfy*

$$\sup_X \varphi = \sup_X \psi = -1.$$

Assume that $H_m(\varphi), H_m(\psi)$ satisfy condition $\mathcal{Q}_m(\alpha, A, \omega)$ for some $\alpha, A > 0$. Then there exists $C = C(\alpha, A, \omega, \|\varphi\|_{L^\infty(X)}, \|\psi\|_{L^\infty(X)}) > 0$ such that, for any $\epsilon > 0$,

$$\sup_X (\psi - \varphi) \leq \epsilon + C[\text{Cap}_{\omega, m}(\varphi - \psi < -\epsilon)]^{\alpha/m}.$$

Proof. The same way as in [18], Proposition 2.6. \square

The following Proposition is due to Kolodziej and Dinew [17], Proposition 2.1. The authors used a tricky argument from complex Monge-Ampère equations in the local setting to produce a comparison between volume and capacity. For convenience we include here a slightly different proof along the same idea.

Proposition 5.4 ([17]). *Let $1 < p < \frac{n}{n-m}$. There exists a constant $C = C(p, \omega)$ such that for every Borel subset K of X , we have*

$$V(K) \leq C \text{Cap}_{\omega, m}(K)^p,$$

where $V(K) := \int_K \omega^n$.

Proof. Fix an open subset U such that $K \subset U$. Solve the complex Monge-Ampère equation to find $u \geq 0$ such that $\omega_u^n = f\omega^n$ on X , with $f = V(U)^{-1}\chi_U$. From [11], Corollary 3.2, the solution u is continuous and moreover, for each $r > 1$,

$$\sup_X u \leq C \|f\|_r^{1/n},$$

where the constant $C = C(r, \omega)$ does not depend on K . The inequality between mixed complex Monge-Ampère measures [16] tells us that

$$\omega_u^m \wedge \omega^{n-m} \geq f^{m/n} \omega^n$$

Thus since $u \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$, we obtain

$$\begin{aligned} \text{Cap}_{\omega, m}(U) &\geq (\sup_X u)^{-m} \int_U H_m(u) \\ &\geq (\sup_X u)^{-m} \int_U f^{m/n} \omega^n \\ &\geq C^{-m} V(U)^{1 - \frac{m}{rn}}. \end{aligned}$$

Thus, for every $r > 1$, there exists a constant C not depending on K such that $V(K) \leq C \cdot \text{Cap}_{\omega, m}(K)^{\frac{nr}{nr-m}}$. The proof is complete. \square

As a consequence of Proposition 5.4 we have some examples of measures satisfying condition $\mathcal{Q}_m(\alpha, A, \omega)$.

Lemma 5.5. *Assume $\mu = f\omega^n$ is a Borel measure with $0 \leq f \in L^p(X)$ for some $p > n/m$. Then for any $0 < \alpha < \frac{mp-n}{(n-m)p}$, there exists $A_\alpha > 0$ such that μ satisfies $\mathcal{Q}_m(\alpha, A_\alpha, \omega)$.*

The following stability theorem was established in [18] in the case of Monge-Ampère equations. With the above tools we can also obtain the same result for complex Hessian equations.

Theorem 5.6. *Assume $H_m(\varphi) = f\omega^n$, $H_m(\psi) = g\omega^n$, where $\varphi, \psi \in \mathcal{P}_m(X, \omega) \cap \mathcal{C}^0(X)$ and $f \in L^p(X)$ with $p > n/m$. Then if γ small enough such that $\frac{\gamma m q}{1-\gamma(1+m q)} < \frac{mp-n}{(n-m)p}$, we have*

$$\|\varphi - \psi\|_{L^\infty(X)} \leq C \|\varphi - \psi\|_{L^1(X)}^\gamma,$$

where $q = p/(p-1)$ denotes the conjugate exponent of p , and the constant C depends only on n, m, p and upper bounds of $\|f\|_p, \|g\|_p$.

Proof. Fix $\epsilon > 0$, and $\alpha > 0$ to be chosen later. It follows from Theorem 5.3 and Proposition 5.2 that

$$\|\varphi - \psi\|_{L^\infty(X)} \leq \epsilon + C_1 [\text{Cap}_{\omega, m}(|\varphi - \psi| > \epsilon)]^{\alpha/m}.$$

Applying Lemma 3.17 we see that

$$\text{Cap}_{\omega, m}(|\varphi - \psi| > \epsilon) \leq \frac{C_2}{\epsilon^{m+1/q}} \int_X |\varphi - \psi|^{1/q} (f + g) \omega^n.$$

It follows thus from Hölder's inequality that

$$\text{Cap}_{\omega, m}(|\varphi - \psi| > \epsilon) \leq \frac{C_3 \|f + g\|_p}{\epsilon^{m+1/q}} \|\varphi - \psi\|_{L^1}^{1/q}.$$

Choose $\epsilon := \|\varphi - \psi\|_{L^1}^\gamma$. Then

$$\text{Cap}_{\omega, m}(|\varphi - \psi| > \epsilon) \leq C_4 [\|\varphi - \psi\|_{L^1}]^{1/q - \gamma(m+1/q)}.$$

We infer that

$$\|\varphi - \psi\|_{L^\infty(X)} \leq \|\varphi - \psi\|_{L^1(X)}^\gamma + C_5 \|\varphi - \psi\|_{L^1(X)}^{\gamma'},$$

where $\gamma' = \frac{\alpha}{m} [1/q - \gamma(m+1/q)]$. We finally choose α so that $\gamma = \gamma'$: this yields the desired estimate. \square

6 Proof of the main result

We first prove the uniqueness. Suppose that φ and ψ are two continuous solutions of (1.1). Set $\rho := \varphi - \psi$. It follows from Corollary 4.4 that

$$\int_X d\rho \wedge d^c \rho \wedge \omega^{n-1} \leq C \left(\int_X \rho (H_m(\psi) - H_m(\varphi)) \right)^{2^{1-m}},$$

where C is a positive constant. Since F is non decreasing in the second variable, it follows from Stokes formula that

$$0 \leq \int_X \rho (H_m(\psi) - H_m(\varphi)) = \int_X (\varphi - \psi) (F(., \psi) - F(., \varphi)) \omega^n \leq 0.$$

Thus,

$$\int_X d\rho \wedge d^c \rho \wedge \omega^{n-1} = 0,$$

which implies that ρ is constant. If moreover $t \mapsto F(x, t)$ is increasing for every $x \in X$, it is easy to see that $\rho = 0$.

Now we prove the existence. We consider three cases.

Case 1: F does not depend on the second variable, $F(x, t) = f(x), \forall x, t$.

Take a sequence of smooth strictly positive functions (f_j) converging to f in $L^p(X)$. We can assume that $\int_X f_j \omega^n = \int_X \omega$, for every j . We use Theorem 2.15 to produce a sequence of smooth solutions (φ_j) normalized by $\sup_X \varphi_j = 0, \forall j$. By passing to a subsequence we can assume that (φ_j) converges in L^1 . Since $\|f_j\|_p$ is uniformly bounded, by Lemma 5.5 we can find α, A not depending on j such that all the measures $f_j \omega^n$ satisfy condition $\mathcal{Q}_m(\alpha, A, \omega)$. By Proposition 5.2 the sequence (φ_j) is uniformly bounded. Now it follows from Theorem 5.6 that φ_j converges uniformly to a function φ which must be continuous. Being the uniform limit (and hence, obviously, quasi-uniform limit) of a smooth sequence in $\mathcal{P}_m(X, \omega)$, the function φ is in the class $\mathcal{P}_m(X, \omega)$ and solves equation $H_m(\varphi) = f \omega^n$.

In the next two cases we will use the Schauder fixed point Theorem.

Case 2: There exists $t_1 \in \mathbb{R}$ such that $\int_X F(x, t_1) \omega^n > \int_X F(x, t_0) \omega^n$.

We set

$$\mathcal{C} := \{\varphi \in SH_m(X, \omega) / \int_X \varphi \omega^n \geq -C_0; \sup_X \varphi \leq 0\}$$

where C_0 is the constant introduced in Lemma 2.14. It follows that \mathcal{C} is a compact convex subset of $L^1(X)$.

Take $\psi \in \mathcal{C}$, we use the result in case 1 to find $\varphi \in \mathcal{P}_m(X, \omega) \cap \mathcal{C}^0(X)$ such that $\sup_X \varphi = 0$ and

$$H_m(\varphi) = F(., \psi + c_\psi)\omega^n,$$

where $c_\psi \geq t_0$ is a constant such that

$$(6.1) \quad \int_X F(., \psi + c_\psi)\omega^n = \int_X \omega^n.$$

This can be done because F satisfies conditions (F2) and (F3). Indeed, by Fatou's Lemma we have

$$\liminf_{t \rightarrow +\infty} \int_X F(., \psi + t)\omega^n \geq \int_X F(., t_1)\omega^n > \int_X \omega^n.$$

Moreover $\int_X F(., \psi + t_0)\omega^n \leq \int_X F(., t_0) = \int_X \omega^n$. Thus by continuity of $t \mapsto \int_X F(., \psi + t)\omega^n$ we can find c_ψ satisfying (6.1).

Observe that φ is well-defined and does not depend on c_ψ . Indeed, assume that c_1, c_2 are two constants such that

$$\int_X F(., \psi + c_1)\omega^n = \int_X F(., \psi + c_2)\omega^n = \int_X \omega^n,$$

and φ_1, φ_2 are two continuous functions in $\mathcal{P}_m(X, \omega)$ such that

$$H_m(\varphi_1) = F(., \psi + c_1), \quad H_m(\varphi_2) = F(., \psi + c_2).$$

Since $t \mapsto F(x, t)$ is non decreasing for every $x \in X$, we have $F(., \psi + c_1) = F(., \psi + c_2)$ almost every where on X . Thus by the uniqueness result above, $\varphi_1 = \varphi_2 + c$ for some constant c which must be 0 by the normalization.

Then we define the map $\Phi : \mathcal{C} \rightarrow \mathcal{C}$, $\psi \mapsto \varphi$.

Now we prove that Φ is continuous on \mathcal{C} . Suppose that (ψ_j) is a sequence in \mathcal{C} converging to $\psi \in \mathcal{C}$ in $L^1(X)$ and let $\varphi_j = \Phi(\psi_j)$. We set $c_j := c_{\psi_j}$ and prove that (c_j) is uniformly bounded. Suppose in the contrary that $c_j \uparrow +\infty$. By subtracting a subsequence if necessary we can assume that $\psi_j \rightarrow \psi$ almost everywhere in X . Then by Fatou's lemma we have

$$\int_X \omega^n = \lim_{j \rightarrow +\infty} \int_X F(., \psi_j + c_j)\omega^n \geq \int_X F(., t_1)\omega^n,$$

which is impossible. Therefore the sequence (c_j) is bounded. This implies that the sequence $(F(., \psi_j + c_j))_j$ is bounded in $L^p(X)$.

To prove the continuity of Φ it suffices to show that any cluster point of (φ_j) satisfies $\Phi(\psi) = \varphi$. Suppose that $\varphi_j \rightarrow \varphi$ in $L^1(X)$. It follows from Theorem 5.6 that the sequence (φ_j) is Cauchy in $\mathcal{C}^0(X)$. Thus (φ_j) converges to φ in $\mathcal{C}^0(X)$ and $\varphi \in \mathcal{P}_m(X, \omega) \cap \mathcal{C}^0(X)$. By subtracting a subsequence if necessary we can assume that $\psi_j \rightarrow \psi$ almost everywhere on X and $c_j \rightarrow c$. Since $t \mapsto F(x, t)$ is continuous we see that $F(., \psi_j + c_j) \rightarrow F(., \psi + c)$ almost everywhere. Thus $H_m(\varphi) = F(., \psi + c)$ which means $\Phi(\psi) = \varphi$ and hence Φ is continuous on \mathcal{C} .

By the Schauder fixed point Theorem, it follows that Φ has a fixed point in \mathcal{C} , say φ . By definition of Φ , the function φ must be in the class $\mathcal{P}_m(X, \omega) \cap \mathcal{C}^0(X)$ and we have

$$H_m(\varphi) = F(., \varphi + c_\varphi)\omega^n.$$

The function $\varphi + c_\varphi$ is the required solution.

Case 3: $\int_X F(., t)\omega^n = \int_X F(., t_0)\omega^n, \forall t \geq t_0$. In this case we have $F(x, t) = F(x, t_0)$ for all $t \geq t_0$ and for almost $x \in X$. Thus

$$\|F(., t_0)\|_{L^p(X)} = \|F(., t)\|_{L^p(X)},$$

for every $t \geq t_0$.

From Proposition 5.2 we can find a positive constant C_1 such that for any $\varphi \in \mathcal{P}_m(X, \omega) \cap \mathcal{C}^0(X)$ satisfying $\sup_X \varphi = 0$ and

$$H_m(\varphi) = f\omega^n,$$

with $\|f\|_p \leq \|F(., t_0)\|_p$ then

$$\varphi \geq -C_1.$$

We set

$$\mathcal{C}' := \{\varphi \in SH_m(X, \omega) / -C_1 \leq \varphi \leq 0\}.$$

Then \mathcal{C}' is a compact convex subset of $L^1(X)$.

Take $\psi \in \mathcal{C}'$, we use the result in case 1 to find $\varphi \in \mathcal{P}_m(X, \omega) \cap \mathcal{C}^0(X)$ such that $\sup_X \varphi = 0$ and

$$H_m(\varphi) = F(., \psi + c_\psi)\omega^n,$$

where $t_0 \leq c_\psi \leq t_0 + C_1$ is a constant such that

$$\int_X F(., \psi + c_\psi)\omega^n = \int_X \omega^n.$$

This can be done because F satisfies the condition (F2) and (F3) . Indeed,

$$\int_X F(., \psi + t_0) \omega^n \leq \int_X \omega^n \leq \int_X F(., \psi + t_0 + C_1) \omega^n.$$

Thus by continuity we can find c_ψ as above.

As in case 2, φ is well-defined and does not depend on the choice of c_ψ . By the choice of C_1 , we see that $\varphi \in \mathcal{C}'$. So we can define a map $\Phi : \mathcal{C}' \rightarrow \mathcal{C}'$ by setting $\Phi(\psi) = \varphi$.

Now we prove that Φ is continuous on \mathcal{C}' . Suppose that (ψ_j) is a sequence in \mathcal{C}' converging to $\psi \in \mathcal{C}'$ in $L^1(X)$ and let $\varphi_j = \Phi(\psi_j)$. We set $c_j := c_{\psi_j}$. For each $j \in \mathbb{N}$,

$$\int_X [F(., \psi_j + c_j)]^p \omega^n \leq \int_X [F(., c_j)]^p \omega^n = \int_X [F(., t_0)]^p \omega^n.$$

Therefore, the sequence $(F(., \psi_j + c_j))_j$ is bounded in $L^p(X)$.

As in case 2, we can assume that $\varphi_j \rightarrow \varphi$ in $L^1(X)$. It follows from Theorem 5.6 that the sequence (φ_j) is Cauchy in $\mathcal{C}^0(X)$. Thus (φ_j) converges to φ in $\mathcal{C}^0(X)$ and $\varphi \in \mathcal{P}_m(X, \omega) \cap \mathcal{C}^0(X)$. By subtracting a subsequence if necessary we can assume that $\psi_j \rightarrow \psi$ in $L^1(X)$ and $c_j \rightarrow c$. Then $H_m(\varphi) = F(., \psi + c)$ and $\Phi(\psi) = \varphi$ which implies that Φ is continuous on \mathcal{C}' .

By the Schauder fixed point Theorem, it follows that Φ has a fixed point in \mathcal{C}' , say φ . By definition of Φ , the function φ must be in the class $\mathcal{P}_m(X, \omega) \cap \mathcal{C}^0(X)$ and we have

$$H_m(\varphi) = F(., \varphi + c_\varphi) \omega^n.$$

The function $\varphi + c_\varphi$ is the required solution.

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